Constraints on the $I = 1$ hadronic τ decay and $e^+e^- \rightarrow hadrons$ data sets and implications for $(g-2)_\mu$

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Sum rule tests are performed on the spectral data for (i) flavor ud vector-current-induced hadronic τ decays and (ii) e^+e^- hadroproduction, in the region below $s \sim 3 - 4$ GeV^2, where discrepancies exist between the isospin-breaking-corrected charged and neutral current $I = 1$ spectral functions. The τ data is found to be compatible with expectations based on high-scale $\alpha_s(M_Z)$ determinations, while the electroproduction data displays two problems. The results favor determinations of the leading order hadronic contribution to $(g-2)_\mu$ which incorporate hadronic τ decay data over those employing electroproduction data only, and hence a reduced discrepancy between experiment and the Standard Model prediction for $(g-2)_\mu$.

I.   INTRODUCTION

In the Standard Model (SM), the largest of the non-purely-leptonic contributions to the anomalous magnetic moment of the muon, $a_\mu \equiv (g-2)_\mu/2$, is that due to the leading order (LO) hadronic vacuum polarization (VP), $[a_\mu]^{LO}_{had}$. a_μ is currently known to 0.5 ppm [1], with a proposal to reduce this to 0.2 ppm in the near future [2]. The 0.5 ppm uncertainty represents $< 1\%$ of $[a_\mu]^{LO}_{had}$, making an accurate determination of $[a_\mu]^{LO}_{had}$ crucial to the study of possible non-SM contributions to a_μ.

$[a_\mu]^{LO}_{had}$ is related to $\sigma\left[e^+e^- \rightarrow hadrons\right]$ by the dispersion representation [3]

$$[a_\mu]^{LO}_{had} = \frac{\alpha^2_{EM}(0)}{3\pi^2} \int_{4m^2_\pi}^{\infty} ds \frac{K(s)}{s} R(s) , \qquad (1)$$

where the form of $K(s)$ is well-known, and $R(s)$ is the "bare" $e^+e^- \rightarrow hadrons$ to $e^+e^- \rightarrow \mu^+\mu^-$ cross-section ratio. With recent electroproduction data, the uncertainty on $[a_\mu]^{LO}_{had}$ from Eq. (1) is comparable to the experimental error on a_μ, and dominates the uncertainty in the SM prediction for a_μ [4, 5, 6, 7, 8]. Since CVC relates the $I = 1$ electromagnetic (EM) spectral function to the charged current vector spectral function

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measured in $\tau^- \to \nu_\tau + $ *non-strange hadrons*, hadronic τ decay data [9, 10, 11] can, in principle, be used to improve the determination of $[a_\mu]_{had}^{LO}$ [12, 13]. At the $< 1\%$ level necessitated by the current experimental error, isospin-breaking (IB) corrections must be taken into account.

IB corrections for the $\pi\pi$ final state, whose contributions dominate $[a_\mu]_{had}^{LO}$, were studied in Refs. [14, 15] and, together with kinematic IB corrections for the 4π contribution, incorporated into the latest τ-based $[a_\mu]_{had}^{LO}$ analyses [5, 7]. Even after IB corrections, however, the high-precision CMD-2 $\pi\pi$ EM data [16] lies $\sim 5-10\%$ below the IB-corrected τ data for $m_{\pi\pi}$ between ~ 0.85 and ~ 1 GeV [5, 17]. The corresponding determinations of $[a_\mu]_{had}^{LO}$ differ by $\sim 2\sigma$, the τ-based result lying higher and producing a SM a_μ prediction in significantly better agreement with experiment [5, 6, 7, 8]. Recent KLOE $e^+e^- \to \pi^+\pi^-$ cross-sections [19] yield an $[a_\mu]_{had}^{LO}$ compatible with CMD-2 [7], though the point-by-point agreement between the two data sets is not entirely satisfactory [7].

In view of the unsettled experimental situation, we study sum rule constraints on weighted integrals of the $I = 1$ vector τ decay and EM spectral functions. The weights, $w(s)$, and upper integration limits, s_0, are chosen such that (i) each spectral integral has a reliable and well-converged OPE representation, and (ii) all relevant OPE input can be obtained from sources independent of the low scale EM and τ data we seek to test. OPE uncertainties are minimized by working with s_0 and $w(s)$ for which the relevant OPE representation is dominated, essentially entirely, by its $D = 0$ component, and hence determined, essentially entirely, by the single input parameter, $\alpha_s(M_Z)$, which can be taken from independent high-scale studies. The τ decay based spectral integrals will be shown to be well reproduced by the corresponding OPE representations. Those based on EM data, in contrast, will be shown to lie consistently below the corresponding OPE values, for positive $w(s)$, and to differ from them in their s_0 dependence. Both features are as expected if the EM spectral data is too low in the disputed region.

II.   THE SUM RULE CONSTRAINTS

We study sum rule constraints on the EM spectral function, $\rho_{EM}(s)$, and the sum of the spin $J = 0$ and 1 components of the charged $I = 1$ vector current spectral function, $\rho_{V;ud}^{(0+1)}(s) \equiv \rho_{V;ud}^{(J=0)}(s) + \rho_{V;ud}^{(J=1)}(s)$. The former is related to $R(s)$ by $\rho_{EM}(s) = R(s)/12\pi^2$, and to the bare $e^+e^- \to$ *hadrons* cross-sections, $\sigma_{bare}(s)$, by

$$\rho_{EM}(s) = s\,\sigma_{bare}(s)/16\pi^3\alpha_{EM}(0)^2 \ . \tag{2}$$

Defining $R_{V;ud}$ by $R_{V;ud} \equiv \Gamma[\tau^- \to \nu_\tau \, \text{hadrons}_{V;ud}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$ and $y_\tau \equiv s/m_\tau^2$, $\rho_{V;ud}^{(0+1)}(s)$ is related to $R_{V;ud}$ by

$$R_{V;ud} = 12\pi^2|V_{ud}|^2 S_{EW} \int_0^1 dy_\tau \, (1-y_\tau)^2 \left[(1+2y_\tau)\,\rho_{V;ud}^{(0+1)}(s) - 2y_\tau\rho_{V;ud}^{(0)}(s)\right] \tag{3}$$

where V_{ud} is the flavor ud CKM matrix element, and S_{EW} is an electroweak correction [21]. Contributions to $\rho_{V;ud}^{(0)}(s)$ are of $O([m_d - m_u]^2)$, and hence numerically negligible, allowing

$\rho_{V;ud}^{(0+1)}(s)$ to be determined from the experimental decay distribution.

A. Finite Energy Sum Rules

For any correlator, $\Pi(s)$, with no kinematic singularities, and any $w(s)$ analytic in $|s| < M$ with $M > s_0$, analyticity implies the finite energy sum rule (FESR) relation

$$\int_0^{s_0} w(s)\,\rho(s)\,ds \;=\; -\frac{1}{2\pi} \oint_{|s|=s_0} w(s)\,\Pi(s)\,ds \;, \tag{4}$$

where $\rho(s)$ is the spectral function of $\Pi(s)$. In QCD, for very large s_0 the OPE representation can be employed on the RHS of Eq. (4). As s_0 is decreased, this representation is expected to break down first near the timelike real s-axis [22]. A range of "intermediate" s_0 will thus exist for which the OPE, though unreliable for general $w(s)$, will remain valid for those $w(s)$ satisfying $w(s = s_0) = 0$. The corresponding FESR's are called "pinched" or pFESR's. For vector (V) and axial vector (A) correlators, and $w(s) = s^N$, OPE breakdown (duality violation) is significant at $s_0 \sim$ a few GeV^2 [23]. Polynomials $w(y)$ (with $y = s/s_0$) having even a single zero at $s = s_0$ ($y = 1$), however, remove such violations for s_0 greater than ~ 2 GeV^2 [23], even for the flavor ud V-A correlator [24].

In interpreting FESR results, one should bear in mind that very strong correlations exist between spectral integrals corresponding to the same $w(y)$, but different s_0. Such correlations are particularly strong when $w(y) \geq 0$ over the relevant interval, $0 < y \leq 1$, and even more so when $w(y)$ is monotonically decreasing. Similar strong correlations exist between spectral integrals corresponding to different $w(y)$, but fixed s_0. Correlations among the corresponding OPE integrals are also very strong, especially when the OPE is dominated, as below, by a single (in this case, $D = 0$) contribution.

We work, in what follows, with pinched polynomial weights, $w(y) = \sum_m c_m y^m$. The pinching condition, $w(1) = 0$ implies $\sum_m c_m = 0$. For reasons explained below, $w(y)$ is further restricted to be non-negative and monotonically decreasing on $0 \leq y \leq 1$. Since s_0 is the only scale entering the RHS of Eq. (4), it is obvious, on dimensional grounds, that integrated OPE contributions of dimension $D = 2k + 2$ scale as $1/s_0^k$, up to logarithms. For $D \geq 2$, such contributions are absent (up to corrections of $O(\alpha_s)$) when $c_{(D-2)/2} = 0$. The structure of the logarithmic integrals, $\oint_{|s|=s_0} ds\, y^k\, \ell n(Q^2/\mu^2)/Q^D$, responsible for the $O(\alpha_s)$ corrections, is such that cancellations inherent in the pinching condition $\sum_m c_m = 0$ lead to strong numerical suppressions of these corrections. $D \geq 8$ contributions are typically assumed to be negligible, since the relevant condensate values are not known phenomenologically. The much stronger s_0 dependence of such high D contributions allows this assumption to be tested explicitly.

The reason for working with non-negative, monotonically decreasing $w(y)$ is that the EM spectral data for the $\pi^+\pi^-$ and $\pi^+\pi^-\pi^0\pi^0$ states, which dominate the EM-τ discrepancy, lie uniformly below the IB-corrected τ data. Non-negativity of $w(y)$ then ensures that, for all s_0, the normalization of the τ-based spectral integrals will be too high if it is the EM data which is correct, while the normalization of the EM spectral integrals will

be too low if it is the τ data which is correct. Since the y value for a given experimental bin decreases with increasing s_0, a monotonically decreasing $w(y)$ similarly ensures that the slope with respect to s_0 of the τ spectral integrals will be too high if the EM data is correct, while the slope with respect to s_0 of the EM spectral integrals will be too low if the τ data is correct. The slope constraint is particularly useful because the slope of the corresponding OPE integrals is very tightly constrained, and only very weakly dependent on the dominant OPE input parameter $\alpha_s(M_Z)$.

B.   OPE Input

Re-writing the weighted pFESR OPE integrals of the relevant correlator, Π, in terms of the Adler function $D(Q^2) \equiv -Q^2 \, d\Pi(Q^2)/dQ^2$, allows potentially large logs in the $D = 0$ contribution to be summed point-by-point along the integration contour. This "contour-improved" (CIPT) prescription is known to improve the convergence of the integrated $D = 0$ series [25]. The Adler function is given by

$$\left[D(Q^2) \right]_{D=0} = C \sum_{k \geq 0} d_k^{(0)} \bar{a}^k \, , \tag{5}$$

where $\bar{a} = a(Q^2) = \alpha_s(Q^2)/\pi$ is the running coupling in the $\overline{MS}$ scheme, and $C = 1$, $2/3$ for the τ, EM cases, respectively. For $N_f = 3$, $d_0^{(0)} = d_1^{(0)} = 1$, $d_2^{(0)} = 1.63982$ and $d_3^{(0)} = 6.37101$ [26]. An estimate for $d_4^{(0)}$, $d_4^{(0)} = 27 \pm 16$ [27] also exists, based on methods which (i) work well for the coefficients of the $D = 0$ series [28] and (ii) produced, in advance of the actual calculation, an accurate prediction for the recently computed $O(a^3)$ $D = 2$ coefficient of the $(J) = (0 + 1)$ V+A correlator sum [29].

The leading $D = 2$ contributions for the τ correlator are $O(m_{u,d}^2)$ and numerically negligible. For the EM correlator, up to tiny $O(m_{u,d}^2/m_s^2)$ corrections, the $D = 2$ contributions are determined by $\bar{a}$ and the running $\overline{MS}$ strange mass $\bar{m}_s$. At the scales employed here the integrated $D = 2$ contribution is small. The full expression for $[\Pi_{EM}(Q^2)]_{D=2}$ may be found in Ref. [30]. The $D = 4$ terms in the OPE of the EM and τ correlators are determined by the RG invariant light quark, strange quark and gluon condensates, $\langle \bar{\ell}\ell \rangle_{RGI}$, $\langle \bar{s}s \rangle_{RGI}$ and $\langle aG^2 \rangle_{RGI}$, up to numerically tiny $O(m_s^4)$ corrections. The expressions may be found in Refs. [31, 32]. The integrated $D = 4$ contributions are again small at the scales employed. To reduce OPE uncertainties, we concentrate here on weights for which the integrated leading $D = 6$ contributions are absent. More extensive studies will be reported elsewhere [33]. We assume throughout that contributions with $D \geq 8$ may be neglected, but check this assumption for self-consistency, as discussed above.

For $D = 4$ input we use (i) $\langle aG^2 \rangle = (0.009 \pm 0.007) \text{ GeV}^4$ (from the recent re-analysis of charmonium sum rules [34]) and (ii) $\langle 2m_\ell \bar{\ell}\ell \rangle = -m_\pi^2 f_\pi^2$ (the GMOR relation). $\langle m_s \bar{s}s \rangle_{RGI}$ then follows from conventional ChPT quark mass ratios and the standard estimate $\langle \bar{s}s \rangle_{RGI}/\langle \bar{\ell}\ell \rangle_{RGI} = 0.8 \pm 0.2$. For the $D = 0$, 2 input, $\bar{a}$ and $\bar{m}_s$, we employ exact solutions based on the 4-loop-truncated β and γ functions [35], with initial conditions

$$m_s(2 \text{ GeV}) = 95 \pm 20 \text{ MeV} \tag{6}$$

$$\alpha_s(M_Z) \ = \ 0.1200 \pm 0.0020. \tag{7}$$

Eq. (6) reflects the range of results obtained in recent sum rule [29, 37] and $N_f = 2$ and $2+1$ unquenched lattice studies [38]. The $N_f = 5$ value in Eq. (7) is run down to the $N_f = 3$ low-scale region using standard 4-loop running and matching [36], with the same matching scales as used in the recent EM sum rule studies of Refs. [6, 20] (HMNT). The input $\alpha_s(M_Z)$ in Eq. (7) differs from the PDG 2004 average for the following reasons. First, the PDG average includes hadronic τ decay input, which must be excluded if we wish to test the τ decay data. Second, the PDG average is strongly affected by the quoted low, small-error determination from heavy quarkonium decay [39]. The Quarkonium Working Group, however, has (i) strongly criticized the input to the low central value, (ii) argued that the quoted error is underestimated by a factor of $3-5$ [41], and (iii) concluded the method is not competitive with extractions based on perturbative treatments of high scale processes [42]. Eq. (7) is obtained by removing lower scale determinations, including those based on heavy quarkonium and τ decay, from the PDG average [43].

C. Spectral Input

The ALEPH and CLEO τ decay distributions are in good agreement. For definiteness, we use the ALEPH results, for which the covariance matrix is publicly available. A small global rescaling accounts for minor changes in B_e, B_μ, and the strange branching fraction, B_s, since the original ALEPH publication [9]. PDG04 [40] values are used for B_e and B_μ, while the updated B_s value incorporates (i) the new (2004) world averages for $B[\tau^- \to \nu_\tau K^- \pi^0]$ and $B[\tau^- \to \nu_\tau K^- \pi^+ \pi^-]$ [44], (ii) the new (2005) CLEO results for the branching fractions of four-particle modes with kaons [45], and (iii) the higher precision $K_{\mu 2}$ value for the K pole contribution [40]. The long-distance EM corrections determined in Ref. [15] are also applied to the dominant $\tau^- \to \pi^- \pi^0$ contribution [15] [46].

Detailed discussions and assessments of the EM hadroproduction data base, as of 2002–2003, can be found in Refs. [4, 5, 6, 47]. The exhaustive compilation of Ref. [47] provides useful information on the treatment of radiative and VP corrections for older experiments where such details are absent from the original publications. Detailed assessments of the needed residual VP corrections are also contained in Refs. [4, 6]. These corrections are computed using the most recent version of F. Jegerlehner's code [48], generously provided by its author. We also employ the following new, high-precision results, published subsequent to the analyses of Refs. [5, 6], and the compilation of Ref. [47]: (i) the final published version of the SND 4π cross-sections [49] (with systematic errors significantly reduced over those of the earlier preprint version); (ii) the updated CMD-2 $\pi^+ \pi^- \pi^+ \pi^-$ [50] and $\pi^0 \gamma$, $\eta \gamma$ [51] cross-sections; and (iii) the BABAR 3π [52] and $\pi^+ \pi^- \pi^+ \pi^-$ [53] cross-sections. Since the $\pi\pi$ component of the EM-τ discrepancy is driven by the CMD-2 data, with its very small 0.6% systematic error [16], we employ only CMD-2 data in the CMD-2 region. Where the existence of newer data permits, we exclude older data for which systematic errors are incompletely stated, or absent, in the original publications, and/or the residual radiative/VP corrections to be applied are unknown. Fortunately, data with missing systematic errors for which no such replacement

is possible play only a small numerical role. We assign a guess of 20% for these errors in such cases. For the small number of remaining older experiments where the situation with regard to residual VP corrections is unclear, we apply no VP correction. In all such cases, however, (i) the corresponding contributions to the spectral integrals are small, and (ii) the neglected VP corrections are, in any case, much less than the quoted systematic errors. The treatment of "missing mode" contributions, and the use of isospin relations for a number of small contributions where direct experimental determinations are absent, or have large errors, follow the treatments discussed in detail in Refs. [4, 6]. More details on the treatment of the EM data will be provided elsewhere [33].

III.   ANALYSIS AND RESULTS

For reasons discussed above, we concentrate on pFESR's involving non-negative, monotonically decreasing $w(y)$. The only such degree 1 weight is $w_1(y) = 1 - y$. Weights with a double zero at $y = 1$, which more strongly suppress OPE contributions from the vicinity of the timelike axis, should be even safer from the point of view of potential duality violation. A convenient set of such "doubly-pinched" weights is the family, $w_N(y) = 1 - \frac{N}{N-1} y + \frac{1}{N-1} y^N$, $N \geq 2$. For a given w_N, the only non-α_s-suppressed ("unsuppressed" in what follows) $D > 4$ OPE contribution is that with $D = 2N + 2$. This contribution scales as $1/s_0^{N+1}$ relative to the leading integrated $D = 0$ term. The strong s_0-dependence allows the neglect of $D \geq 8$ contributions (for $w_{N \geq 3}$) to be tested for self-consistency. We have also studied pFESR's based on a number of other weights. Since the results in all cases point to the same conclusion, and OPE uncertainties are reduced for weights having no unsuppressed $D = 6$ contribution, we focus on the pFESR's for two of the weights defined above, w_1 and w_6. Other results will be presented elsewhere [33].

Combined OPE errors for the various pFESR's are obtained by adding in quadrature uncertainties associated with the OPE input parameters ($D = 4$ condensates, m_s, and $\alpha_s(M_Z)$) and the truncation/residual scale dependence of the integrated $D = 0$, 2 series. The latter are estimated to be *twice* the magnitude of the last term in the corresponding truncated series. The resulting OPE error estimate is somewhat more conservative than that employed by HMNT.

Results for the EM case are presented in Figures 1, 2, those for the τ case in Figures 3, 4. The dashed lines represent the central OPE results, the solid lines the upper and lower edges of the OPE error bands. Because of the strong correlations, the OPE band is better thought of as a bundle of allowed parallel lines than as a generally allowed region. We see that, in the region 2 GeV$^2 < s_0 < m_\tau^2$, both the magnitude and slope of the integrals over the τ decay distribution are in good agreement with OPE expectations. In contrast, the EM spectral integrals are consistently low relative to OPE expectations (particularly for s_0 greater than $\sim$ 2.5 GeV^2) and have slopes with respect to s_0 significantly lower than those of the OPE curves. Both features of the EM results are as expected if the τ data is correct, and hence the EM data low, in the disputed region.

The implications of the normalizations of the EM and τ spectral integrals can be quantified by working out the $\alpha_s(M_Z)$ required to match the OPE and spectral sides of

TABLE I: The values of $\alpha_s(M_Z)$ obtained by fitting to the experimental EM and τ spectral integrals for $s_0 = m_\tau^2$, with central values for the $D = 2, 4$ OPE input

Weight	$[\alpha_s(M_Z)]_{EM}$	$[\alpha_s(M_Z)]_\tau$
w_1	$0.1138^{+0.0030}_{-0.0035}$	$0.1212^{+0.0027}_{-0.0032}$
w_6	$0.1150^{+0.0022}_{-0.0026}$	$0.1195^{+0.0020}_{-0.0022}$

a given pFESR. The reliability of the OPE, and hence of the extraction of $\alpha_s(M_Z)$, is optimized by choosing s_0 as large as possible – in the case of hadronic τ decay, $s_0 = m_\tau^2$. Results corresponding to central input for the small $D = 2, 4$ OPE contributions and the $s_0 = m_\tau^2$ values of the spectral integrals are shown in Table I for both the EM and τ cases. The agreement between the τ-decay and independent high-scale determinations is excellent [54]. In contrast, the EM data corresponds to $\alpha_s(M_Z) \sim 2\sigma$ lower than the high-scale determination. The y^6 term of $w_6(y)$, in principle, produces an unsuppressed $D = 14$ OPE contribution scaling as $1/s_0^6$ ($1/s_0^7$ relative to the leading $D = 0$ term). Such a contribution, if present, would contaminate the extraction of $\alpha_s(M_Z)$. There is, however, no evidence for such a contribution, at a level which would impact our analysis, in the s_0 dependence of either the EM or τ w_6-weighted spectral integrals [55]. The excellent agreement between the $\alpha_s(M_Z)$ extracted using different doubly-pinched weights, with potential unsuppressed $D > 6$ contributions of different dimension, provides further evidence in support of the absence of such $D > 6$ contributions [56, 57].

To quantify the disagreement between the EM OPE and experimental slope values, we work out the correlated errors for the slopes with respect to s_0 of the OPE and spectral integrals. The correlations are such that the uncertainty on the OPE side is rather small. In particular, the slope is quite insensitive to $\alpha_s(M_Z)$. These points are illustrated in Table II, which shows the spectral integral and OPE slope values for the EM w_1 and w_6 pFESR's. The OPE entries labelled "indep" are those obtained using the independent, high-scale fit value for $\alpha_s(M_Z)$. Those labelled "fit", in contrast, correspond to the $\alpha_s(M_Z)$ values obtained by fitting to the relevant $s_0 = m_\tau^2$ spectral integrals, as given in Table I. We see that, even if one were willing to tolerate the lower central $\alpha_s(M_Z)$ values implied by the EM spectral integrals, such a lowering of $\alpha_s(M_Z)$ would have negligible impact on the OPE vs. spectral integral slope discrepancy problem.

TABLE II: Slopes wrt s_0 of the EM OPE and spectral integrals

Weight	S_{exp}	$\alpha_s(M_Z)$	S_{OPE}
w_1	$.00872 \pm .00026$	indep	$.00943 \pm .00008$
		fit	$.00934 \pm .00008$
w_6	$.00762 \pm .00017$	indep	$.00811 \pm .00009$
		fit	$.00805 \pm .00009$

IV. DISCUSSION AND CONCLUSIONS

We have shown that weighted spectral integrals constructed using $I = 1$ hadronic τ decay data are in good agreement with OPE expectations, while those involving EM data (i) require a value of $\alpha_s(M_Z) \sim 2\sigma$ below that given by high-scale determinations and (ii) correspond to a slope with respect to s_0 in $\sim 2.5\sigma$ disagreement with the OPE prediction. The slope problem, moreover, *cannot* be cured simply by adopting the lower $\alpha_s(M_Z)$ values, shown in Table I, which would bring the normalization of the OPE and spectral integrals into agreement for $s_0 \simeq m_\tau^2$. The insensitivity of the slope to $\alpha_s(M_Z)$ also means that the agreement between the OPE expectation and the observed slope for the τ decay spectral integrals represents a non-trivial test of the τ data.

One possibility is that the problems with the EM sum rules might be attributable to the presence of residual duality violation at the intermediate scales studied here; the success of the OPE in predicting both the slope and magnitude of the τ-decay-based spectral integrals over the whole of the region $2 \text{ GeV}^2 < s_0 < m_\tau^2$, however, renders such an explanation highly implausible. The results thus point to the reliability of the τ data, and to the likelihood of either (i) a problem with the experimental EM spectral distribution, or (ii) the presence of as-yet-unidentified non-one-photon physics contributions in the experimental EM cross-sections. This in turn suggests that a_μ determinations which incorporate τ decay data are to be favored over those employing EM spectral data only.

While the disagreement between the EM and high-scale determinations of $\alpha_s(M_Z)$ is only $\sim 2\sigma$, even with significantly lower high-scale input, e.g., the 2002 PDG average $\alpha_s(M_Z) = 0.1172 \pm 0.0020$ used by HMNT, the EM normalization, and even more so the EM slope, would still require, on average, upward fluctuations in $\rho_{EM}(s)$. Since $K(s)/s > 0$, such fluctuations would typically also increase $[a_\mu]_{had}^{LO}$. Thus, even ignoring the slope problem and assessing the EM data as moderately consistent, within errors, with the OPE constraints, the fact that the spectral integrals lie consistently below the corresponding OPE constraint values, for any sensible input $\alpha_s(M_Z)$, points to the likelihood of a $[a_\mu]_{had}^{LO}$ value higher than the current central EM-data-based value.

Two further points are of relevance to assessing the implications of our results for the value of $[a_\mu]_{had}^{LO}$. First, we find that, replacing the EM $\pi^+\pi^-$, $\pi^+\pi^-\pi^0\pi^0$ and $\pi^+\pi^-\pi^+\pi^-$ data with the equivalent τ data resolves completely both the normalization and slope problems for the resulting modified "EM" spectral integrals [58]. Second, it is readily demonstrated that the pFESR's employed are sensitive to, not just the discrepancies in the 4π region, but also those in the 2π region. (This is relevant since the $[a_\mu]_{had}^{LO}$ integral is dominated by the 2π spectral contribution.) In fact, for the w_1 pFESR, the shift in the EM spectral integral associated with the modification of the 2π part of the EM spectral function represents 82% of the full shift at $s_0 = 2 \text{ GeV}^2$ and 32% at $s_0 = m_\tau^2$. The corresponding figures for the w_6 pFESR are 87% at $s_0 = 2 \text{ GeV}^2$ and 45% at $s_0 = m_\tau^2$. Thus, even though the pFESR's employed are relatively more sensitive to the 4π spectral contributions than is the $[a_\mu]_{had}^{LO}$ integral, a clear sensitivity to the 2π component remains, making the constraints associated with these pFESR's highly relevant to the $[a_\mu]_{had}^{LO}$ problem.

In conclusion, all the sum rule tests performed favor the reliability of the τ decay data, and point to problems with the EM data. We conclude that, at present, determinations of $[a_\mu]_{had}^{LO}$ employing IB-corrected τ decay data are more reliable than those based on EM data alone, and hence that there is no clear sign of a discrepancy between the current experimental value for a_μ and the SM prediction.

NOTE ADDED: Subsequent to the submission of this paper, new results for the $e^+e^- \to \pi^+\pi^-$ cross-sections were released by the SND Collaboration [59]. As would be expected from the sum rule results above, the SND cross-sections are compatible with the $\tau \to \nu_\tau \pi\pi$ data, but in significant disagreement with the KLOE $\pi\pi$ data, in the disputed region.

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[55] An extremely conservative upper bound on the impact of such a term can be obtained by adding a $D = 14$ contribution to the OPE side and performing a fit for its strength, ignoring the very strong correlations between w_6-weighted integrals at different s_0. In the τ case, the resulting $D = 14$ contribution is $< 5\%$ of the $D = 0$ contribution at $s_0 = 2$ GeV^2, and hence $< 0.2\%$ at $s_0 = m_\tau^2$. Incorporating such a contribution would shift the extracted value of $\alpha_s(M_Z)$ by < 0.0003. Accounting for the strong correlations would lead to an even smaller shift. Note also that it is impossible to attribute the EM slope and normalization problems to neglected higher D contributions: a $D = 14$ term with strength sufficient to bring the w_6-weighted EM OPE and spectral integrals into agreement at $s_0 \simeq 4$ GeV^2 would be ~ 4 times the corresponding $D = 0$ contribution at $s_0 = 2$ GeV^2, and produce a disastrously bad match between the OPE and spectral integral curves.

[56] The $\alpha_s(M_Z)$ values extracted using the w_3, w_4, and w_5 sum rules are, for the EM case, $0.1152^{+0.0019}_{-0.0021}$, $0.1154^{+0.0020}_{-0.0023}$, and $0.1152^{+0.0022}_{-0.0024}$, respectively, and, for the τ case, $0.1189^{+0.0018}_{-0.0021}$, $0.1193^{+0.0019}_{-0.0022}$, and $0.1194^{+0.0020}_{-0.0022}$, respectively. In both cases, the results are

in excellent agreement with those obtained using the corresponding w_6 sum rule.

[57] The ALEPH [9] and OPAL [10] collaborations have, in fact, performed extractions of the effective $D = 6, 8$ condensate combinations for the ud V correlator by fitting to the $s_0 = m_\tau^2$ spectral integrals for a number of different "spectral weights". The reader is cautioned that these extractions are performed assuming $D > 8$ contributions are absent, despite the fact that, for all but one of the weights employed, such contributions are, in principle, present in unsuppressed form. The analogous assumption for the ud V-A correlator has been tested explicitly and found to be incompatible with data [24], making it also suspect for the V case. One can nonetheless take the extracted values as representative of what might be expected for the size of such effects and investigate the impact on the sum rules used in this analysis. By construction, no unsuppressed $D = 6$ contributions are present in the sum rules considered. While the $D = 8$ term does not contribute to the w_1 or w_6 pFESR's, it *does* contribute to the w_3 pFESR. Using the larger of the ALEPH and OPAL $D = 8$ fitted values, it is straightforward to demonstrate that, at $s_0 = m_\tau^2$, the resulting $D = 8$ contribution is at the level of 0.2% of the dominant $D = 0$ term, producing a shift in the value of $\alpha_s(M_Z)$ extracted using the w_3 sum rule of < 0.0003, which, as claimed, is much smaller than the $\sim .0020$ error associated with data uncertainties. This observation simply confirms the conclusion obtained by studying the s_0 dependence of the various sum rules and the consistency of the $\alpha_s(M_Z)$ extractions from sum rules where non-α_s-suppressed contributions with different $D > 6$ would be present [56].

[58] The normalizations of the modified EM integrals are now in full agreement with the high-scale-input OPE constraint bands, for all $s_0 > 2$ GeV^2. The modified spectral integral slopes are also in good agreement with OPE expectations (the modified central values are 0.00947 for the w_1 and 0.00807 for the w_6 pFESR, to be compared with the OPE predictions 0.00943 ± 0.0008 and 0.00811 ± 0.00009 already reported in Table II.

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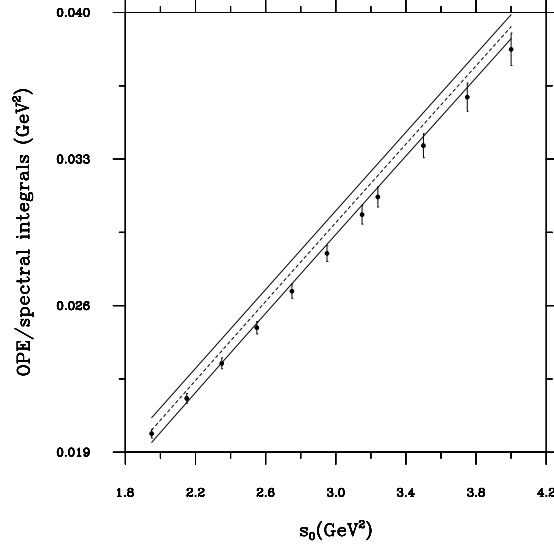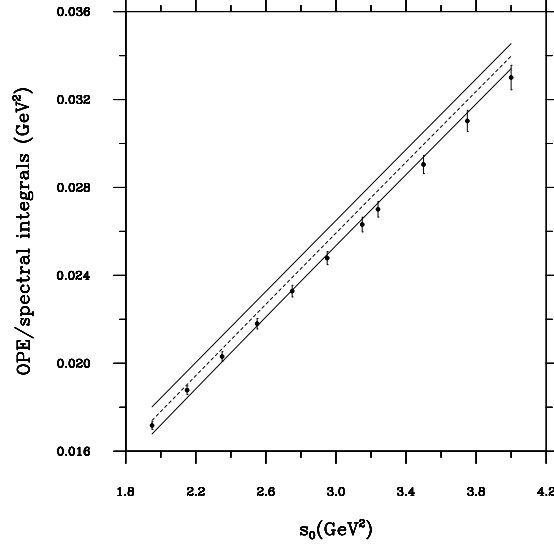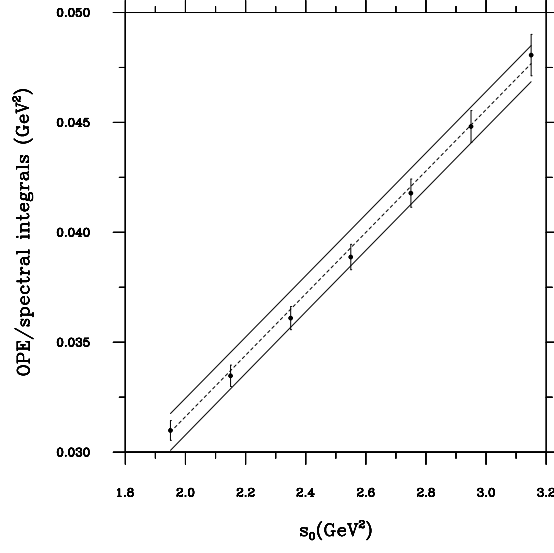
FIG. 1: EM OPE and spectral integrals for the weight w_1

FIG. 2: EM OPE and spectral integrals for the weight w_6

FIG. 3: τ OPE and spectral integrals for the weight w_1

FIG. 4: τ OPE and spectral integrals for the weight w_6